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The number of spanning forests of a graph

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Abstract

In this paper we study the number of spanning forests of a graph. Let G be a connected simple graph. (1) We give a lower bound for the number of spanning forests of G in terms of the edge connectivity of G. (2) We give an upper bound for the number of rooted spanning forests of G. (3) We describe the elementary symmetric functions of inverse positive Laplacian eigenvalues of a tree. (4) We determine all Laplacian integral graphs with prime number of spanning trees. (5) We give a simple proof of a theorem of K. Hashimoto on Ihara zeta function.

Keywords: Graph spectrum; Graph Laplacian; Forest

1. Introduction

For any graph G with no loops, and no multiple edges, let $V=V(G)$ and $E=E(G)$ be the vertex set and the edge set of G, respectively. For a graph G, we denote the number of vertices by $|G|$. Let $A(G)$ be the adjacency matrix of G, and $D(G)=\mathrm{diag}(\deg(v))_{v\in V(G)}$ the degree matrix of G. Then the matrix $L(G)=D(G)-A(G)$ is called the *Laplacian matrix* of the graph G. The *Laplacian eigenvalues* of a graph are defined to be the eigenvalues of its Laplacian matrix.

For a graph G of order n, let

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n = 0$$

be its Laplacian eigenvalues of G (repeated according to their multiplicities).

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Then by the matrix-tree theorem [2, Corollary 6.5], the number $\kappa(G)$ of the spanning trees of a graph G of order n is given by

$$\kappa(G) = \frac{\lambda_1\lambda_2\cdots\lambda_{n-1}}{n}.$$

A subgraph of G is called a *spanning forest* of G, if it contains no cycle and all vertices in G. In Proposition 2.1, we shall give a lower bound for the number of spanning forests of G with given number of connected components in terms of the edge connectivity of G. In Proposition 2.4, we shall give an upper bound for the minimum number of spanning forests with k connected components which contains given k vertices in different component trees. In Theorem 2.7, we shall give a formula for the elementary symmetric function of the inverse positive Laplacian eigenvalues of a tree.

A graph G is called a *Laplacian integral graph* if all Laplacian eigenvalues are integers. It is known (cf. [8,9]) that a tree is Laplacian integral if and only if it is a star graph. In section three, we shall determine all Laplacian integral graphs with prime number of spanning trees.

Let $Z_G(u)$ be the Ihara zeta function associated with a connected graph G of order n and size m. In [6], Hashimoto proved that the value of $(1-u)^{n-m-1}Z_G(u)$ at $u=1$ is equal to $2^{m-n+1}(n-m)\kappa(G)$. In Section 4, we shall give a proof of Hashimoto's theorem.

2. Bounds for the number of spanning forests

We denote by $\{T_1,\ldots,T_k\}$ a spanning forest of a graph G with connected components $T_1,\ldots,T_k$, and the set of spanning forests of G with k connected components will be denoted by $\mathrm{For}_G(k)$. We denote by $\delta(T_1,\ldots,T_k)$ the set of edges whose end vertices are contained in distinct component trees.

Proposition 2.1. *Let G be a connected graph of order n with edge connectivity $\lambda(G)$. Then*

$$|\mathrm{For}_G(k)| \geqslant \left(\frac{\lambda(G)}{2}\right)^{n-k}\binom{n}{k}.$$

Proof. Let $\{T_1,\ldots,T_{k+1}\}$ be a spanning forest in $\mathrm{For}_G(k+1)$. Note that, for each i, $1 \leqslant i \leqslant k+1$, the number of edges e with exactly one end vertex of e is contained in T_i is at least $\lambda(G)$. Thus we have

$$\frac{\lambda(G)(k+1)}{2} \leqslant |\delta(T_1,\ldots,T_{k+1})|. \tag{2.1}$$

Add one edge in $\delta(T_1,\ldots,T_{k+1})$ to $\{T_1,\ldots,T_{k+1}\}$. Then we obtain a spanning forest in $\mathrm{For}_G(k)$. Thus for a given forest in $\mathrm{For}_G(k+1)$, by inequality (2.1), we obtain at least $\lambda(G)(k+1)/2$ forests in $\mathrm{For}_G(k)$. On the other hand, from a spanning forest $\{T_1,\ldots,T_k\}$, delete one edge in it. Then since $\{T_1,\ldots,T_k\}$ has $n-k$ edges, we obtain $n-k$ forests in $\mathrm{For}_G(k+1)$. Consequently by the double counting principle,

$$\frac{\lambda(G)(k+1)}{2}\,|\mathrm{For}_G(k+1)| \leqslant (n-k)|\mathrm{For}_G(k)|.$$

Thus we have

$$\frac{\lambda(G)(k+1)}{2}\,|\mathrm{For}_G(k+1)|\leqslant(n-k)|\mathrm{For}_G(k)|$$

$$\frac{\lambda(G)(k+2)}{2}\,|\mathrm{For}_G(k+2)|\leqslant(n-k-1)|\mathrm{For}_G(k+1)|$$

$$\cdots \qquad \cdots$$

$$\frac{\lambda(G)n}{2}\,|\mathrm{For}_G(n)|\leqslant|\mathrm{For}_G(n-1)|.$$

Since, $|\mathrm{For}_G(n)|=1$, these inequalities yield the desired result. $\square$

Corollary 2.2. *Let G be a connected graph of order n with edge connectivity $\lambda(G)$. Then*

$$\kappa(G)\geqslant\left(\frac{\lambda(G)}{2}\right)^{n-1}n.$$

Let A be an arbitrary k-set of the vertices in G. We denote by $\mathrm{For}_G^A(k)$ the set of spanning forests of G with k component trees where the vertices of A appear in different trees.

Lemma 2.3. *Let G be a connected graph of order n with positive Laplacian eigenvalues $\lambda_1,\ldots,\lambda_{n-1}$. Then*

$$\sum_{A\in\binom{V}{k}}|\mathrm{For}_G^A(k)|=e_{n-k}(\lambda_1,\ldots,\lambda_{n-1}),$$

where $e_{n-k}(\lambda_1,\ldots,\lambda_{n-1})$ denotes the elementary symmetric polynomial of degree $n-k$ in $\lambda_1,\ldots,\lambda_{n-1}$.

Proof. For a given spanning forest $\{T_1,\ldots,T_k\}\in\mathrm{For}_G(k)$, selecting one vertex from each component tree, we obtain a k-set A of vertices. Thus we obtain

$$\sum_{A\in\binom{V}{k}}|\mathrm{For}_G^A(k)|=\sum_{\{T_1,\ldots,T_k\}\in\mathrm{For}_G(k)}|T_1|\cdots|T_k|.$$

By Theorem 7.5 in [2],

$$e_{n-k}(\lambda_1,\ldots,\lambda_{n-1})=\sum_{\{T_1,\ldots,T_k\}\in\mathrm{For}_G(k)}|T_1|\cdots|T_k|.$$

This completes the proof. $\square$

Proposition 2.4. *Let G be a connected graph of order n and size m. For a positive integer k, $1\leqslant k\leqslant n$, let*

$$\kappa_k(G):=\min_{A\in\binom{V}{k}}|\mathrm{For}_G^A(k)|.$$

Then

$$\kappa_k(G) \leqslant \frac{k}{n}\left(\frac{2m}{n-1}\right)^{n-k}, \tag{2.2}$$

where equality holds if and only if G is the complete graph K_n of order n.

Proof. By Lemma 2.3, $\kappa_k(G) \leqslant e_{n-k}(\lambda_1, \ldots, \lambda_{n-1}) / \binom{n}{k}$, and hence, since

$$\binom{n-1}{n-k} = \frac{k}{n}\binom{n}{n-k},$$

we have

$$\kappa_k(G) \leqslant \frac{k}{n} e_{n-k}(\lambda_1, \ldots, \lambda_{n-1}) \Big/ \binom{n-1}{n-k}.$$

On the other hand, by Maclaurin's inequality [5, Theorem 52] on elementary symmetric functions,

$$\frac{e_{n-k}(\lambda_1, \ldots, \lambda_{n-1})}{\binom{n-1}{n-k}} \leqslant \left(\frac{e_1(\lambda_1, \ldots, \lambda_{n-1})}{\binom{n-1}{1}}\right)^{n-k} = \left(\frac{2m}{n-1}\right)^{n-k},$$

thus we have proved inequality (2.2).

In inequality (2.2), equality holds if and only if $\lambda_1 = \cdots = \lambda_{n-1}$, and $|\mathrm{For}_G^A(k)|$ is constant on $\binom{V}{k}$. As is easily shown this condition is satisfied if and only if G is the complete graph K_n. □

The following result is due to Grimmett [3].

Corollary 2.5. *Let G be a connected graph of order n and size m. Then*

$$\kappa(G) \leqslant \frac{1}{n}\left(\frac{2m}{n-1}\right)^{n-1}.$$

Proof. Since $\kappa(G) = \kappa_1(G)$, the result follows from Proposition 2.4. □

For vertices u, v of a connected graph G, denote by $d_G(u, v)$ the distance between u and v. The *Wiener index* $W(G)$ of G is defined by

$$W(G) = \sum_{\{u,v\} \in \binom{V}{2}} d_G(u, v).$$

Lemma 2.6. *For a spanning forest* $\{T_1, \dots, T_k\}$ *of a connected graph G, set*

$$w(T_1, \dots, T_k) = \sum_{i=1}^{k} \frac{W(T_i)}{|T_i|}.$$

Then

$$\sum_{\{T_1,\dots,T_k\}\in \mathrm{For}_G(k)} w(T_1, \dots, T_k)|T_1| \cdots |T_k| = \sum_{\{T_1,\dots,T_{k+1}\}\in \mathrm{For}_G(k+1)} |\delta(T_1, \dots, T_{k+1})||T_1| \cdots |T_{k+1}|.$$

Proof. For a given $\mathscr{T} = \{T_1, \dots, T_{k+1}\} \in \mathrm{For}_G(k+1)$, we can select $|T_1| \cdots |T_{k+1}|$ $k+1$ -sets A in V such that the vertices appear in different component trees, and so we have

$$\sum_{\mathscr{T}\in \mathrm{For}_G(k+1)} |\delta(\mathscr{T})||T_1| \cdots |T_{k+1}| = \sum_{A\in\binom{V}{k}} \sum_{\mathscr{T}\in \mathrm{For}_G^A(k+1)} |\delta(\mathscr{T})||T_1| \cdots |T_{k+1}|,$$

where $\delta(\mathscr{T}) = \delta(T_1, \dots, T_{k+1})$.

For a $k+1$-set A of the vertices, denote by $\mathrm{For}_G^A(k)$ the set of spanning forests of G with k component trees where the vertices of A appear in different trees except one vertex in A. For a given forest $\mathscr{T} \in \mathrm{For}_G^A(k+1)$, adding one edge from $\delta(\mathscr{T})$, we obtain a spanning forest in $\mathrm{For}_G^A(k)$. Conversely, for a given forest $\{T_1, \dots, T_k\} \in \mathrm{For}_G^A(k)$, if a tree T_i in $\{T_1, \dots, T_k\}$ contains two vertices, say u and v, in A delete one edge from the u-v path of length $d_T(u, v)$ in T_i. Then we obtain a forest in $\mathrm{For}_G^A(k+1)$.

Consequently, by the double counting principle, we obtain

$$\sum_{A\in\binom{V}{k}} \sum_{\mathscr{T}\in \mathrm{For}_G^A(k+1)} |\delta(\mathscr{T})||T_1| \cdots |T_{k+1}| = \sum_{\{T_1,\dots,T_k\}\in \mathrm{For}_G(k)} w(T_1, \dots, T_k)|T_1| \cdots |T_k|,$$

completing the proof. □

Theorem 2.7. *Let T be a tree of order n with positive Laplacian eigenvalues* $\lambda_1, \dots, \lambda_{n-1}$. *Then*

$$e_k\left(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_{n-1}}\right) = \frac{1}{kn} \sum_{\{T_1,\dots,T_k\}\in \mathrm{For}_T(k)} w(T_1, \dots, T_k)|T_1| \cdots |T_k|.$$

Proof. Since T is a tree, by the matrix-tree theorem, $n = \lambda_1 \cdots \lambda_{n-1}$. Hence e_{n-k-1} $(\lambda_1, \dots, \lambda_{n-1}) = ne_k(\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_{n-1}})$, and $\delta(T_1, \dots, T_{k+1}) = k$. Then by Lemma 2.3, the right-hand side of Lemma 2.6 is equal to

$$
\begin{aligned}
&= k \sum_{\{T_1,\ldots,T_{k+1}\} \in \mathrm{For}_T(k+1)} |T_1| \cdots |T_{k+1}| \\
&= k e_{n-k-1}(\lambda_1, \ldots, \lambda_{n-1}) \\
&= (kn) e_k \left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_{n-1}} \right),
\end{aligned}
$$

completing the proof. □

The following beautiful result is due to McKay (cf. [10]):

Corollary 2.8. *Let T be a tree of order n with positive Laplacian eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$. Then*

$$
W(T) = n \left(\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_{n-1}} \right).
$$

3. Laplacian integral graphs with prime $\kappa(G)$

A tree is Laplacian integral if and only if it is a star graph. Thus, in general, there are infinitely many Laplacian integral graphs with given number of spanning trees. A connected graph G is said to be *primitive* if its complement G^c is also connected. In ([12, Theorem 3.7]), it is shown that there are only finitely many primitive Laplacian integral graphs with fixed number of spanning trees.

On the other hand, if λ is a multiple Laplacian integral eigenvalue of a connected graph G, then $\kappa(G)$ is divisible by λ ([12, Theorem 1.4]). Thus, if the number of spanning trees of a Laplacian integral graph G has relatively few divisors, then the number of multiple Laplacian eigenvalues is also small. This imposes a strong condition on the structure of G. In such cases, we can expect to determine all primitive Laplacian integral graphs.

In this section, we shall determine all Laplacian integral graphs with prime number of spanning trees.

For two graphs G_1 and G_2, denote by $G_1 * G_2$ the graph obtained by adding all possible edges (u, v), $u \in V(G_1)$ and $v \in V(G_2)$. The graph $G_1 * G_2$ is called the *join* of G_1 and G_2. Recall that a graph G is Laplacian integral if and only if its complement G^c is Laplacian integral. Then, we have

Lemma 3.1. *If a graph G is the join of two graphs $G = G_1 * G_2$, then G is Laplacian integral if and only if G_1 and G_2 are Laplacian integral.*

For a graph G, let $\lambda_{\max}$ and $d_{\max}$ denote the maximum Laplacian eigenvalue and the maximum degree of G, respectively. In [4] Grone and Merris have shown the following result:

Lemma 3.2. *If G has at least one edge, then*

$$\lambda_{\max} \geqslant d_{\max} + 1. \tag{3.3}$$

Moreover if G is a connected graph of order $n \geqslant 2$, *equality holds in* (3.3) *if and only if* $d_{\max} = n - 1$.

Theorem 3.3. *Let G be a connected Laplacian integral graph of order* $n \geqslant 3$ *with* $\kappa(G) = p$, *where* p *is a prime number. Then* $p = 3$, *and G has a vertex u of degree* $n - 1$. *Moreover in* $V(G) - \{u\}$, *there are two vertices of degree* 2 *and, if* $n > 3$, *other vertices have degree* 1.

Proof. Let

$$\lambda_1 = \lambda_{\max} \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{n-1}$$

be positive Laplacian eigenvalues of G.

Case 1: the edge connectivity $\lambda(G) \geqslant 2$.

If the edge connectivity $\lambda(G) \geqslant 2$, then by Corollary 2.2, $p = \kappa(G) \geqslant n$, and hence

$$n = \frac{\lambda_1 \cdots \lambda_{n-1}}{p} \leqslant p. \tag{3.4}$$

Since p is a prime number and G is Laplacian integral, a Laplacian eigenvalue is divisible by p. This implies that

$$p \geqslant n \geqslant \lambda_{\max} \geqslant p$$

and so, we have

$$p = n = \lambda_{\max}. \tag{3.5}$$

By (3.4) and (3.5), we have

$$\lambda_1 = \lambda_2 = p, \quad \lambda_3 = \cdots = \lambda_{n-1} = 1. \tag{3.6}$$

By (3.6), the complement G^{c} of G has three connected components. If $n \geqslant 4$, since $n - 1$ is a Laplacian eigenvalue of a connected component of G^{c} and each connected component of G^{c} has at most $n - 2$ vertices, we have $n - 1 \leqslant n - 2$, a contradiction. Therefore in this case, $G = K_3$, the complete graph of order 3.

Case 2: the edge connectivity $\lambda(G) = 1$.

If $\lambda(G) = 1$, G has a *bridge*. Since $\kappa(G)$ is a prime number, this implies that G has a vertex u of degree one. Assume now that G^{c} is connected. Since the maximum degree of G^{c} is $n-2$, and G^{c} is Laplacian integral, it follows from Lemma 3.2 that the maximum Laplacian eigenvalue of G^{c} is equal to n. But this implies that G is not connected, a contradiction. Therefore G^{c} is not connected, and hence G is a join of two graphs:

$$G = G_1 * G_2, \quad |G_1| \leqslant |G_2|.$$

Since the vertex u has degree one, $V(G_1) = \{v\}$, for some $v \in V(G)$, and $u \in V(G_2)$. Then by Lemma 3.1, G is Laplacian integral if and only if $G - u$ is Laplacian integral.

Let $S = \{v_1, \ldots, v_s\}$ be the set of vertices of G with degree one. Then repeating this procedure, we see that G is Laplacian integral if and only if $G - S$ is Laplacian integral. Since $\kappa(G - S) = \kappa(G)$ and $\kappa(G - S)$ has no vertex of degree one, it follows from *Case* 1 that $G - S$ is the complete graph of order 3. This completes the proof. □

4. Ihara zeta function

In this section, we shall give a new proof of a result of Hashimoto on the Ihara zeta function associated with a connected graph.

The Ihara zeta function $Z_G(u)$ associated with a connected graph G is a function of a sufficiently small complex variable u defined by

$$Z_G(u) = \prod_{[C]} (1 - u^{l(C)})^{-1},$$

where the product is over all equivalent classes of primitive closed backtrackless, tail-less cycles C of positive length, and $l(C)$ denotes the length of C.

Ihara [7] proved that if G is a regular graph, $Z_G(u)$ is expressed as the reciprocal of a polynomial involving the adjacency matrix $A(G)$.

Ihara's theorem has been generalized to non-regular graphs (cf. [1,11]):

Theorem 4.1 (*Bass*). *For a connected graph G of order n and size m,*

$$Z_G^{-1}(u)(1 - u^2)^{n-m} = \det(I - (L(G) - D(G))u + (D(G) - I)u^2).$$

In [6], Hashimoto proved the following theorem. We give a proof of Hashimoto's theorem.

Theorem 4.2 (*Hashimoto*). *Let G be a connected graph of order n and size m.*

$$Z_G(u)^{-1}(1 - u)^{n-m-1}|_{u=1} = 2^{m-n+1}(n - m)\kappa(G).$$

Proof. Let $\lambda_1, \ldots, \lambda_{n-1}$ be positive Laplacian eigenvalues of G. Since $L(G)$ is a real symmetric matrix, it is diagonalized by an orthogonal matrix K:

$$\mathrm{diag}(0, \lambda_1, \ldots, \lambda_{n-1}) = K^t L(G) K.$$

Since 0 is a simple eigenvalue of $L(G)$ and $j = (1, \ldots, 1)^t$ is an eigenvector of $L(G)$ with respect to 0, we can take $j/\sqrt{n}$ as the first column of the matrix K.

Then using Theorem 4.1, the matrix tree theorem and the relation

$$2m = \sum_{v \in V(G)} \deg(v),$$

we obtain

$$
\begin{aligned}
&-2^{n-m}Z_G(u)^{-1}(1-u)^{n-m-1}|_{u=1}\\
&\quad=\lim_{u\to 1}\frac{Z_G(u)^{-1}(1-u^2)^{n-m}}{u-1}\\
&\quad=\lim_{\varepsilon\to 0}\det(L(G)+\varepsilon(-2I+L(G)+D(G)))/\varepsilon\\
&\quad=\lim_{\varepsilon\to 0}\det(\mathrm{diag}(0,\lambda_1,\ldots,\lambda_{n-1})+\varepsilon K^t(-2I+L(G)+D(G))K)/\varepsilon\\
&\quad=\left(-2+\frac{j^t}{\sqrt{n}}D(G)\frac{j}{\sqrt{n}}\right)\lambda_1\cdots\lambda_{n-1}\\
&\quad=\left(-2n+\sum_{v\in V(G)}\deg(v)\right)\frac{\lambda_1\cdots\lambda_{n-1}}{n}\\
&\quad=2(m-n)\kappa(G),
\end{aligned}
$$

completing the proof. □

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